# ON THE REDUCTION OF A BILINEAR QUANTIC OF THE $n^{\text{TH}}$ ORDER TO THE FORM OF A SUM OF $n$ PRODUCTS BY A DOUBLE ORTHOGONAL SUBSTITUTION.

By Professor *Sylvester*.

A HOMOGENEOUS lineo-linear function in two sets of variables

$$x, y, \ldots z;\ u, v, \ldots w$$

will contain $n^2$ terms: two independent orthogonal substitutions performed on the two sets will introduce twice $\frac{1}{2}n(n-1)$ disposable constants, and by a suitable choice of these, $n^2-n$ terms of the transformed function may be made to vanish so as to leave a sum of products of the new $x, y, \ldots z$ paired with the new $u, v, \ldots w$: it will of course be found in general impossible to obliterate *any* arbitrarily chosen $(n^2-n)$ terms in the transformed function; since if in the $n$ remaining products one letter of one set were combined with more than one of the other set, this would (by means of a further superimposed orthogonal substitution) be equivalent to taking away more than $(n^2-n)$ terms by means of only $(n^2-n)$ disposable constants. It is very easy to effect the transformation indicated by a method very analogous to that of reducing a quadric in $n$ variables by an orthogonal substitution to its canonical form, and to show *à posteriori* that the substitutions are always, real in this case as in the other, when the original coefficients are real; but it will, I think, (although not necessary) be found interesting and instructive to prove *à priori* the latter assertion by a similar method to that applied to Quadrics in the last number of the *Messenger*. I will begin then with this proof, reserving the complete solution of the problem to the end of the article. The leading idea in this as in the preceding article is to regard a finite orthogonal substitution as the product of an infinite number of infinitesimal ones.

1°. For $\qquad axu + \alpha xv + \beta yu + byv.$

Let $x, y;\ u, v$ become $x+\epsilon y,\ -\epsilon x+y;\ u+\lambda v,\ -\lambda u+v$ respectively, then

$$\delta\alpha = a\lambda - b\epsilon, \quad \delta\beta = a\epsilon - b\lambda,$$

$$\alpha\delta\alpha + \beta\delta\beta = (a\alpha - b\beta)\lambda + (a\beta - b\alpha)\epsilon.$$

Hence $\alpha^2+\beta^2$ may be made to decrease unless $a=0$, $b=0$, or $\alpha=0$, $\beta=0$, or $\dfrac{a}{b}=\dfrac{\alpha}{\beta}=\pm 1$, in which case since

$$(a+\alpha)(x+y)(u+v)+(a-\alpha)(x-y)(u-v)$$
$$=2a(xu+yv)+2\alpha(xv+yu),$$
$$(a-\alpha)(x+y)(u-v)+(a+\alpha)(x-y)(u+v)$$
$$=2a(xu-yv)+2\alpha(xv-yu),$$

the form is immediately canonizable.

Hence in the infinite succession of infinitesimal orthogonal substitutions (equivalent to a single one) either $a$ and $b$ or $\alpha$ and $\beta$ must vanish simultaneously, on which supposition the form is canonical or else it is reducible to the canonical form by a second finite orthogonal substitution.

Let us now proceed to the case of a *ternary* bilinear form in $x, y, z$; $u, v, w$.

I suppose by the previous case the form to be deprived of two terms, and that we have to deal with the form

$$axu+byv+fxw+guz+hyw+kvz+czw.$$

*Lemma.* If $f=0$, $g=0$, or $h=0$, $k=0$ the above form is reducible by the previous case. Also if $a^2=b^2$ and $f=0$, $h=0$, or $g=0$, $k=0$, or $a^2=b^2$ and $\left(\dfrac{f}{h}\right)^2=\left(\dfrac{g}{k}\right)^2$ the form is reducible to the previous case by a single additional finite orthogonal transformation.

For the sake of brevity I leave the proof to my readers.

Introducing now two infinitesimal orthogonal substitutions with parameters $\varepsilon, \eta, \theta$; $\lambda, \mu, \nu$,* we obtain the variations

$$\delta f=a\mu-h\varepsilon-c\eta,\quad \delta h=b\nu+f\varepsilon-c\theta,$$
$$\delta g=a\eta-k\lambda-c\mu,\quad \delta k=b\theta+g\lambda-c\nu,$$

also in order to keep the coefficients of $xv$, $yu$ at *null*, we must have

$$a\lambda-b\varepsilon-f\nu-k\eta=0,$$
$$-b\lambda+a\varepsilon-g\theta-h\mu=0.$$

* The positive values of the parameters in each system are supposed to belong to the upper, and the negative values to the lower half of each orthogonal matrix.

From the previous equations we obtain

$$f\delta f + g\delta g + h\delta h + k\delta k$$
$$= (af - cg)\,\mu + (bh - ck)\,\nu + (ag - cf)\,\eta + (bk - ch)\,\theta.$$

1°. Suppose $a^2 - b^2$ *not* zero; then $\mu$, $\nu$, $\eta$, $\theta$ will be independent and their coefficients cannot all become zero unless $f^2 = g^2$ and $h^2 = k^2$, or else $f = 0$ and $g = 0$, or $h = 0$ and $k = 0$, on either of which suppositions the form becomes canonizable by virtue of the Lemma.

2°. Let $a^2 = b^2$. Then we must have

$$f\nu + k\eta \pm (g\theta + h\mu) = 0,$$

which I shall satisfy by making $f\nu \pm g\theta = 0$, $k\eta \pm h\mu = 0$.

Hence

$$\Sigma f\delta f = \{(af - cg)\,k \mp (ag - cf)\,h\}\,\rho + \{(ah - ck)\,g \mp (ak - ch)\,f\}\,\tau,$$

$\rho$, $\tau$ being two arbitrary infinitesimals.

Therefore $\Sigma f\delta f$ may be made negative unless the multipliers of $\rho$ and $\tau$ are both zero, in which case by addition or subtraction we obtain $fk = gh$; consequently two out of the four variables $f$, $g$, $h$, $k$ are zero, or else $\dfrac{f}{h} = \dfrac{g}{k}$, and on either of these suppositions the transformed function may be canonized by virtue of what has been proved in the case of two biliteral sets, or may by a finite orthogonal substitution be brought to a form so canonizable.

Hence it is clear that either $f$, $g$, $h$, $k$ may all be made to vanish, or else we must pass through a form known to be canonizable. This is the proof for a bilinear function of triliteral sets, which may be easily extended to a bilinear function of $n$-literal sets.

I will now give the method for effecting the reduction which is thus proved to be always capable of being effected by real substitutions.

Let $\Sigma a_{r,s} x_r y_s$ be the given bilinear function $B$.

Then $\Sigma \left(\dfrac{dB}{dy_s}\right)^2$ which is an orthogonal invariant of $B$ quâ the $y$'s, is a Quadratic function of the $x$'s, which will have an orthogonal substitute *quâ* the $x$'s of the form $\Sigma\,[\lambda_r x_r^2]$.

If then $B$ is reducible by a double orthogonal substitution to the form $\Sigma\,[\theta_r x_r y_r]$, we must have $\Sigma\,[\theta_r x_r]^2$ orthogonally equivalent to $\Sigma\,[\lambda_r x_r^2]$, and this can only be the case when the $\theta$'s are respectively (in any order) the squares of the $\lambda$'s.

The $\theta$'s I call the Canonical Multipliers to $B$.

This gives rise to the following rule:

Form the Matrix $[m]$.

$$\begin{matrix} a_{1,1}, & a_{2,1} \dots a_{n,1}, \\ a_{1,2}, & a_{2,2} \dots a_{n,2}, \\ \dots\dots & \dots\dots\dots\dots, \\ a_{1,n}, & a_{2,n} \dots a_{n,n}. \end{matrix}$$

From this derive a Matrix $[M]$, a *false* square of $[m]$, obtained by multiplying each line in it by *all* the lines (according to Cauchy's rule, in fact, for the multiplication of *Determinants*). Then the latent roots of $[M]$ are the squares of the Canonical Multipliers to $B$.

But if instead of $\Sigma\left(\frac{dB}{dy_s}\right)^2$ we take $\Sigma\left(\frac{dB}{dx_r}\right)^2$ and deal with it in like manner, we shall obtain a matrix $[n]$, such that $[m]$ and $[n]$ are transverse to each other, the lines and columns of the one being the columns and lines of the other: the Cauchian Square of $[n]$ will give rise to a matrix $[N]$ different from $[M]$ but having the same latent roots: in fact the coefficients of the equation to the latent roots alike of $m$ and of $[n]$ with the signs in the alternate places changed will be unity, the sum of the squares of all the terms in $[m]$ or $[n]$, the sum of the squares of the minors of the 2^rd^, 3^rd^, ... orders in $[m]$ or $[n]$; and finally the last coefficient will be the square of the determinant to $[m]$ or $[n]$: so that we shall obtain as we ought the same set of canonical multipliers whichever matrix $[M]$ or $[N]$ we employ; but in order to obtain the substitutions which must be impressed on the $x$ set and the $y$ set to arrive at the Canonical form in which only $n$ products appear we shall want both $[M]$ and $[N]$. Let me, however, pause for a moment to call attention to the interesting fact that the sum of the squares of the coefficients in $B$ by virtue of being a coefficient of the latent function to $[M]$ or $[N]$ is necessarily a bi-orthogonal invariant to $B$; so, too, all the other coefficients in this function are such invariants: and among them the last, which is the square of the determinant to $[m]$ or $[n]$. Thus then this determinant (which may be termed the discriminant) is an invariant alike for the two theories; viz. the better known one in which the $x$ set and the $y$ set are subjected to the same general substitution, and the one here considered where these sets are subjected to two independent orthogonal substitutions.

In either theory the vanishing of the discriminant is the signal of the Canonical form becoming short of one term.

It is also proper to notice that the latent roots of $[M]$ or $[N]$, which by virtue of $[M]$ and $[N]$ being symmetrical matrices are necessarily real, are for these particular forms of $[M]$ and $[N]$ *positive* as well as real since the coefficients with the alternate signs changed are all positive, being the sums of squares of real numbers.

To complete the solution it remains to find the two canonizing orthogonal matrices, but these are known by the ordinary theory for quadrics: thus the $x$ substitution will be that which canonizes $[M]$ and the $y$ substitution that which canonizes $[N]$.

Conversely, if $[M]$ and $[N]$ are supposed given, we shall know the linear functions of the $x$'s which substituted for $x_1, x_2, \ldots x_n$ and the linear function of the $y$'s which substituted for $y_1, y_2, \ldots y_n$, such that $\Sigma\lambda_1^{\frac{1}{2}}x_1y_1$ shall be identical with $B$, the $\lambda$'s being the latent roots common to $[M]$ and $[N]$. There will be $2^n$ systems of values represented by $\lambda_1^{\frac{1}{2}}, \lambda_2^{\frac{1}{2}}, \ldots \lambda_n^{\frac{1}{2}}$: thus then $2^n$ matrices transverse to one another can be found such that their *false* squares shall be respectively identical with any two given symmetrical matrices having the same latent roots, and we are thus enabled indirectly, through the theory of bi-orthogonal canonization, to obtain the solution of a problem which intrinsically has or seems to have nothing to do with orthogonal or other transformation.

It is worthy of observation that this problem of finding the so-to-say *false* square root common to two given symmetrical matrices having the same latent equation, admits of precisely the same number ($2^n$) solutions as the problem of finding the true square root of one general matrix. For if $[M]$ be any given matrix of order $n$ and $[1]$ represents the unit matrix of that order, namely the matrix all of whose terms are zeros except those in the principal diagonal which are units, we know by virtue of a general theorem that calling $\lambda_1, \lambda_2, \ldots \lambda_n$ its $n$ latent roots, each true square root of $[M]$ is represented by

$$\Sigma\lambda_1^{\frac{1}{2}}\frac{([M]-\lambda_2[1])([M]-\lambda_3[1])\ldots([M]-\lambda_n[1])}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\ldots(\lambda_1-\lambda_n)}.$$

New College, Oxford,
*5th April*, 1889.